THE EFFECTIVE CONE OF THE SPACE OF PARAMETRIZED RATIONAL CURVES IN A GRASSMANNIAN

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ABSTRACT. We determine the effective cone of the Quot scheme parametrizing all rank r, degree d quotient sheaves of the trivial bundle of rank n on $\mathbb{P}^1$. More specifically, we explicitly construct two effective divisors which span the effective cone, and we also express their classes in the Picard group in terms of a known basis.

1. INTRODUCTION

Let V be an n-dimensional vector space over an algebraically closed ground field, and let $V_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1} \otimes V$ be the trivial bundle of rank n on $\mathbb{P}^1$. In this paper, we determine the effective cone of the Quot scheme R parametrizing all degree d, rank r quotient sheaves of $V_{\mathbb{P}^1}$.

The Quot scheme R is closely related to the Kontsevich moduli spaces of genus-zero stable maps to Grassmannians. Indeed let $\mathbb{G}$ be the Grassmannian of r-dimensional quotient spaces of V. Then R is a compactification of $R^0 = \mathrm{Mor}_d(\mathbb{P}^1, \mathbb{G})$, the space of all degree d morphisms from $\mathbb{P}^1$ to $\mathbb{G}$, or equivalently, the space of all degree d, rank r quotient bundles of $V_{\mathbb{P}^1}$. One can also consider Quot schemes over more general curves other than $\mathbb{P}^1$, and like the Kontsevich moduli spaces, they have been used in enumerative geometry (see for example [1], [2], [3], [7], [5], [6]).

Many basic properties of the Quot scheme R were established in [9] by Strømme. For example he showed that R is an irreducible, rational, nonsingular projective variety of dimension $nd + r(n-r)$ [9, Theorem 2.1]. It is not too difficult to see that if $r = n-1$ then R is a projective space, and if $d = 0$ then R is just the Grassmannian $\mathbb{G}$, so in these cases $\mathrm{Pic}\, R \cong \mathbb{Z}$ [9, Proposition 6.1]. In all other cases, i.e. $0 \le r \le n-2$ and $d \ge 1$, Strømme proved that $\mathrm{Pic}\, R \cong \mathbb{Z}^2$, and he gave generators for the nef cone of R [9, Theorem 6.2]. More specifically, there is a universal short exact sequence on $R \times \mathbb{P}^1$:

$$0 \to \mathcal{A} \to V_{R \times \mathbb{P}^1} \to \mathcal{B} \to 0.$$

For each point $p \in R$, $\mathcal{B}_p = \mathcal{B}|_{\{p\} \times \mathbb{P}^1}$ is the degree d, rank r quotient sheaf of $V_{\mathbb{P}^1}$ represented by p. Denote by π_1 and π_2 the two projection maps from $R \times \mathbb{P}^1$ to R and $\mathbb{P}^1$, respectively. Let $\mathcal{B}(m) = \mathcal{B} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(m)$, and let $B_m = \pi_{1*} \mathcal{B}(m)$. Then the

divisor classes $c_1(B_{d-1})$ and $c_1(B_d) - c_1(B_{d-1})$ on R form a $\mathbb{Z}$-basis for Pic R and also span the nef cone.

There is another useful $\mathbb{Z}$-basis for Pic R described by Ramirez in [6, §3].[1] Let $h = c_1\big(\pi_2^* \mathcal{O}_{\mathbb{P}^1}(1)\big)$. Then the divisor classes

$$Y = \pi_{1*}\big(h \cdot c_1(\mathcal{B})\big), \quad D = \pi_{1*}\big(c_2(\mathcal{B})\big)$$

form a $\mathbb{Z}$-basis for Pic R. Indeed a Grothendieck-Riemann-Roch computation [6, Lemma 3.2] yields[2]

$$c_1(B_d) - c_1(B_{d-1}) = Y,$$
$$c_1(B_{d-1}) = 2dY - D.$$

Thus in terms of the basis Y and D, the nef cone is spanned by Y and $2dY - D$.

The divisors D and Y have geometric interpretations, at least when $r \geq 2$. To see this, let $\mathbb{P}(V)$ be the projective space of one-dimensional subspaces of V. Each point $p \in R^0$ can be viewed as a parametrized rational scroll $\mathbb{P}(\mathcal{A}_p) \to \mathbb{P}(V)$ of dimension $n - r$ in $\mathbb{P}(V)$. Hence those points in R^0 which, when viewed as rational scrolls, meet a fixed $(r-2)$-dimensional subspace in $\mathbb{P}(V)$, form a divisor of R^0, and the closure of this divisor in R is D (note that it was shown in [8, Corollary 3.3.8] that the complement of R^0 in R is irreducible of codimension r). Similarly, those points in R^0 for which, when viewed as rational scrolls, the fibers over a fixed point $0 \in \mathbb{P}^1$ meet a fixed $(r-1)$-dimensional subspace in $\mathbb{P}(V)$, form a divisor of R^0, and Y is the closure of this divisor in R.

In order to describe the effective cone of R, we construct two effective divisors D_{unb} and D_{deg}, and then use test curves to show that they span the effective cone, as well as to express their classes in Pic R in terms of the basis Y and D. The definitions of D_{unb} and D_{deg} were inspired by the work of Coskun-Starr [4] on the effective cone of the Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(\mathbb{G}, d)$. They did not have a complete description of the cone when $r < d$, but they showed that if $n - r$ and d are fixed, then the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{G}, d)$ grows as r increases, and stabilizes when $r \geq d$. They then went on to show that the stable effective cone, i.e. the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{G}, d)$ when $d = r$, is spanned by the boundary divisors and two more effective divisors which they named D_{unb} and D_{deg} (the subscripts stand for "unbalanced" and "degenerate", respectively). To state the definitions, let $k = n - r$ be the rank of the universal subsheaf $\mathcal{A}$. The definition of D_{unb} depends on whether k divides d. When $k \mid d$, we define D_{unb} in the same way as Coskun-Starr:

$D_{\mathrm{unb}} = \{p \in R \mid \mathcal{A}_p$ has unbalanced splitting (as a locally free sheaf on $\mathbb{P}^1)\}$.

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[1] Although there was a standing assumption that $n = 4$ and $r = 2$ throughout [6], the part in §3 about changing basis for Pic R can be straightforwardly generalized.

[2] There is a sign error in the second formula in [6, Lemma 3.2]. Except for that, both formulae again hold true for any $n \geq 2$, $0 \leq r \leq n - 2$, and $d \geq 1$.

When $k \nmid d$, set $d = kd_1 + (k - \ell_1)$ for $0 < \ell_1 < k$. Then for all p outside a codimension two locus in R, $\mathcal{A}_p$ splits as a direct sum of ℓ_1 copies of $\mathcal{O}_{\mathbb{P}^1}(-d_1)$ and $(k - \ell_1)$ copies of $\mathcal{O}_{\mathbb{P}^1}(-d_1 - 1)$. In this situation the direct sum of the ℓ_1 copies of $\mathcal{O}_{\mathbb{P}^1}(-d_1)$ is a distinguished subsheaf $\mathcal{E}_p$ of $\mathcal{A}_p$, whose projectivization $\mathbb{P}(\mathcal{E}_p) \to \mathbb{P}(V)$ is the directrix of the scroll $\mathbb{P}(\mathcal{A}_p) \to \mathbb{P}(V)$. We define D_{unb} to be the closure in R of the locus of p whose corresponding directrix $\mathbb{P}(\mathcal{E}_p) \to \mathbb{P}(V)$ meets a fixed $(n - 2 - \ell_1)$-dimensional subspace in $\mathbb{P}(V)$. Note that this is different from the definition of Coskun-Starr in [4], which asked that the *linear span* of the directrix meets a fixed subspace in $\mathbb{P}(V)$ of codimension $\ell_1(d_1 + 1)$. We modified it because it does not work when $d \gg r$.

Coskun-Starr defined D_{deg} on $\overline{\mathcal{M}}_{0,0}(\mathbb{G}, d)$, $d = r \geq 2$, to be the divisor of maps whose corresponding scrolls degenerate, namely lie in some hyperplane. This does not give a divisor as soon as $d > r$, so new approaches are required to formulate a definition of D_{deg} on R which will work for all $d \geq 1$. Our starting point is the observation that in the case $d = r \geq 2$ considered by Coskun-Starr, if $p \in R^0$, then the scroll corresponding to the subbundle $\mathcal{A}_p$ is degenerate if and only if the quotient bundle $\mathcal{B}_p$ has unbalanced splitting (Proposition 4.1). Inspired by this, whenever $r \geq 2$ we define D_{deg} to be the closure in R of the locus of those $p \in R^0$ whose corresponding quotient bundles $\mathcal{B}_p$ are "unbalanced in the previously defined sense", which means unbalanced splitting if $r \mid d$, or incidence condition on the directrix of the dual scroll $\mathbb{P}(\mathcal{B}_p^{\vee}) \to \mathbb{P}(V^{\vee})$ if $r \nmid d$ (details are spelled out in Section 4, the paragraph following Proposition 4.1). When $r = 1$ however, no p in R^0 has unbalanced quotient bundle $\mathcal{B}_p$; on the other hand, the complement of R^0 in R becomes an irreducible divisor by [8, Corollary 3.3.8]. Therefore it is natural to define $D_{\mathrm{deg}} = R \setminus R^0$ when $r = 1$. Finally when $r = 0$, $\mathcal{B}_p$ is a torsion sheaf of degree d for every $p \in R$. Generically the support of $\mathcal{B}_p$ consists of d distinct points on $\mathbb{P}^1$, and we define D_{deg} to be the locus of $p \in R$ for which some of the d points in $\mathrm{Supp}\,\mathcal{B}_p$ coincide.

With all the necessary definitions in place, we can now state the main theorem.

Theorem 1.1. *Let R be the Quot scheme parametrizing all rank r, degree d quotient sheaves of the trivial bundle of rank n on $\mathbb{P}^1$. Let $k = n - r$ and assume that $k \geq 2$ and $d \geq 1$. Then the effective cone of R is spanned by the two effective divisors D_{unb} and D_{deg} defined above. Moreover, their classes in $\mathrm{Pic}\,R$ can be expressed in terms of the basis D and Y as*

$$D_{\mathrm{unb}} = c_1 \left(-D + \left(d + \left\lceil \frac{d}{k} \right\rceil \right) Y \right),$$

$$D_{\mathrm{deg}} = \begin{cases} c_2 \left(D + \left(-d + \left\lceil \frac{d}{r} \right\rceil \right) Y \right), & \text{if } r > 0; \\ 2(d - 1)Y, & \text{if } r = 0, \end{cases}$$

for some positive c_1 and c_2. Precisely, $c_1 = 1$ if $k \mid d$ and $r \neq 0$, while $c_1 = d_1(\ell_1 + 1)$ if $k \nmid d$ and we set $d = kd_1 + (k - \ell_1)$ for $0 < \ell_1 < k$. Similarly $c_2 = 1$ if $r \mid d$, while $c_2 = d_2(\ell_2 + 1)$ if $r \nmid d$ and we set $d = rd_2 + (r - \ell_2)$ for $0 < \ell_2 < r$.

We do not know the value of c_1 when $r = 0$ and $k \mid d$, although the natural guess is that it is 1. The nef cone and the effective cone of R are shown in Figure 1.
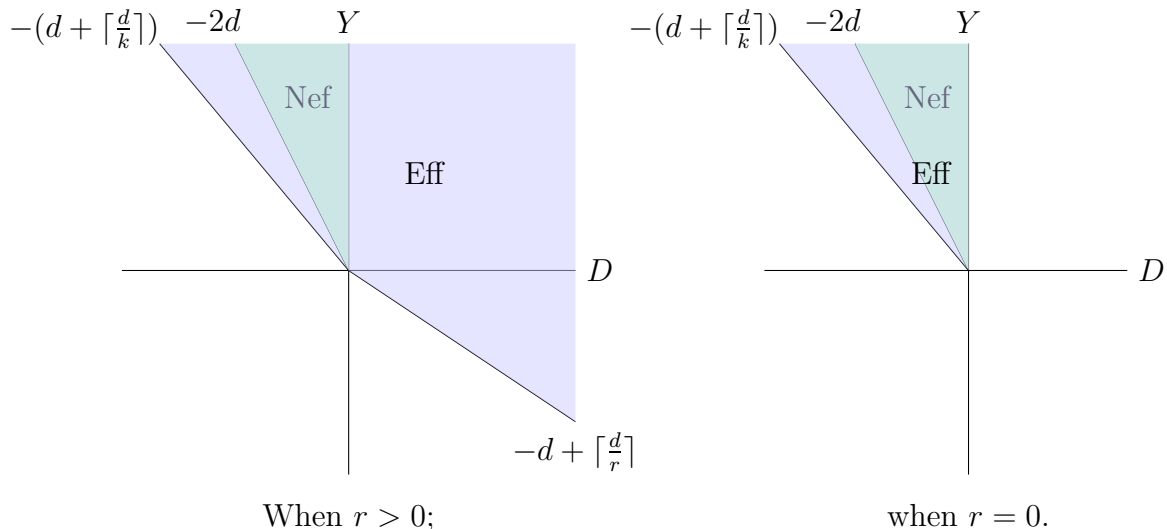

FIGURE 1. The nef cone and the effective cone of R. Left: $r > 0$. Right: $r = 0$. The slopes of the boundaries of the cones are labeled.

In order to prove Theorem 1.1, we start with constructing test curves on R in Section 2. Then we compute the intersection numbers of the test curves with the divisors D and Y in Section 3, and with D_{unb} and D_{deg} in Section 4. These intersection numbers come into the proof of Theorem 1.1 given in Section 5 (the case $r > 0$) and Section 6 (the case $r = 0$). It is perhaps worth mentioning that we use a somewhat nonstandard Lemma 5.1 to show that the two divisors D_{unb} and D_{deg} span the effective cone, which avoids the task of constructing moving curves in the usual strategy.

2. TEST CURVES ON THE QUOT SCHEME

In this section, we define the test curves we will be using to intersect with the various divisors on the Quot scheme R. To this end, pick an arbitrary smooth projective curve C and k line bundles $A_1, \ldots, A_k$ on C with nonzero sections. Let $a_i = \deg A_i \geq 0$ for $i = 1, \ldots, k$. Let $S = C \times \mathbb{P}^1$, and let $\pi_1 \colon S \to C$ and $\pi_2 \colon S \to \mathbb{P}^1$ be the two

projection maps. If L is a line bundle on C and $m \in \mathbb{Z}$, we denote by $L(m)$ the line bundle $\pi_1^* L \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(m)$ on S. Let A_S be the direct sum of the line bundles

$$A_S = \bigoplus_{i=1}^{k} A_i^{\vee}(-m_i)$$

on S, where if $k \mid d$ then $m_i = d/k$ for all i, and if $k \nmid d$, set $d = kd_1 + (k - \ell_1)$ for $0 < \ell_1 < k$, and choose $m_i = d_1$ for $i \le \ell_1$ and $m_i = d_1 + 1$ for $i > \ell_1$. (In other words, $m_1 + \cdots + m_k = d$ is the "most balanced" partition of d into k parts.) Given a section s_i of $A_i(m_i) \otimes V$ for each $i = 1, \ldots, k$, one obtains a sheaf homomorphism

$$\varphi \colon A_S = \bigoplus_{i=1}^{k} A_i^{\vee}(-m_i) \longrightarrow V_S = \mathcal{O}_S \otimes V,$$

which is simply multiplication by s_i on each summand $A_i^{\vee}(-m_i)$. If a_i's are sufficiently large and s_i's are sufficiently general, the restriction of φ on each fiber of π_1 is injective, so its cokernel is a degree d, rank r quotient sheaf of $V_{\mathbb{P}^1}$. Hence φ induces a morphism

$$\alpha \colon C \longrightarrow R,$$

which is one of the test curves we will use.

When $r > 0$, we can construct another curve

$$\beta \colon C \longrightarrow R$$

which is induced by a surjective sheaf homomorphism

$$\psi \colon V_S \longrightarrow B_S = \bigoplus_{i=1}^{r} B_i(n_i),$$

where B_i are line bundles of degrees $b_i \ge 0$ on C, and $n_i = d/r$ for all i if $r \mid d$, while if $r \nmid d$, write $d = rd_2 + (r - \ell_2)$ for $0 < \ell_2 < r$, and choose $n_i = d_2$ for $i \le \ell_2$ and $n_i = d_2 + 1$ for $i > \ell_2$. (In other words, $n_1 + \cdots + n_r = d$ is the "most balanced" partition of d into r parts.) Since we always have $k \ge 2$, such a surjective homomorphism ψ exists as long as $b_i = \deg B_i$ are sufficiently large.

When $r = 0$, we can no longer construct the curve β as above, so a separate construction is needed. This will be done in Section 6.

3. Intersections of the test curves with D and Y

In this section, we compute the intersection numbers $\alpha \cdot D$, $\alpha \cdot Y$, $\beta \cdot D$, and $\beta \cdot Y$, where α and β are the test curves defined in Section 2, and D and Y are the divisor classes which form a $\mathbb{Z}$-basis for $\mathrm{Pic}\, R$ described in the Introduction.

Recall that if

$$0 \to \mathcal{A} \to V_{R \times \mathbb{P}^1} \to \mathcal{B} \to 0$$

is the universal short exact sequence on $R \times \mathbb{P}^1$, and $h = c_1\big(\pi_2^* \mathcal{O}_{\mathbb{P}^1}(1)\big)$, then

$$Y = \pi_{1*}\big(h \cdot c_1(\mathcal{B})\big), \quad D = \pi_{1*}\big(c_2(\mathcal{B})\big).$$

By Whitney's formula,

$$c_1(\mathcal{B}) = -c_1(\mathcal{A}), \quad c_2(\mathcal{B}) = c_1(\mathcal{A})^2 - c_2(\mathcal{A}).$$

So we also have

$$Y = \pi_{1*}\big(-h \cdot c_1(\mathcal{A})\big), \quad D = \pi_{1*}\big(c_1(\mathcal{A})^2 - c_2(\mathcal{A})\big).$$

Recall that the curve $\alpha \colon C \to R$ is induced by an injective sheaf homomorphism $\varphi \colon A_S \to V_S$ on $S = C \times \mathbb{P}^1$, where

$$A_S = \bigoplus_{i=1}^{k} A_i^{\vee}(-m_i).$$

Hence

$$\alpha \cdot Y = -h \cdot c_1(A_S) = \sum_{i=1}^{k} a_i\,;$$

$$\alpha \cdot D = c_1(A_S)^2 - c_2(A_S) = 2d \sum_{i=1}^{k} a_i - \sum_{1 \le i \ne j \le k} a_i m_j$$

$$= \begin{cases} (d + d_1) \displaystyle\sum_{i=1}^{k} a_i, & \text{if } k \mid d; \\[2em] (d + d_1) \displaystyle\sum_{i=1}^{\ell_1} a_i + (d + d_1 + 1) \displaystyle\sum_{i=\ell_1+1}^{k} a_i, & \text{if } k \nmid d, \end{cases}$$

where $a_i = \deg A_i$ for all i, $d_1 = \lfloor d/k \rfloor$, and $\ell_1 = k(d_1 + 1) - d$.

The computation for the other curve β is similar. Recall that $\beta \colon C \to R$ is induced by a surjective sheaf homomorphism $\psi \colon V_S \to B_S$ on $S = C \times \mathbb{P}^1$, where

$$B_S = \bigoplus_{i=1}^{r} B_i(n_i).$$

Hence

$$\beta \cdot Y = h \cdot c_1(B_S) = \sum_{i=1}^{r} b_i \, ;$$

$$\beta \cdot D = c_2(B_S) = \sum_{1 \leq i \neq j \leq r} b_i n_j = \begin{cases} (d - d_2) \displaystyle\sum_{i=1}^{r} b_i, & \text{if } r \mid d; \\[2em] (d - d_2) \displaystyle\sum_{i=1}^{\ell_2} b_i + (d - d_2 - 1) \displaystyle\sum_{i=\ell_2+1}^{r} b_i, & \text{if } r \nmid d, \end{cases}$$

where $b_i = \deg B_i$ for all i, $d_2 = \lfloor d/r \rfloor$, and $\ell_2 = r(d_2 + 1) - d$.

4. Intersections of the test curves with D_{unb} and D_{deg}

In this section, we compute the intersection numbers $\alpha \cdot D_{\mathrm{unb}}$, $\alpha \cdot D_{\mathrm{deg}}$, $\beta \cdot D_{\mathrm{unb}}$, and $\beta \cdot D_{\mathrm{deg}}$, where α and β are the test curves defined in Section 2, and D_{unb} and D_{deg} are the effective divisors described in the Introduction which are to be shown to span the effective cone of R. The answers are summarized in Proposition 4.3 at the end of the section.

Let

$$0 \to \mathcal{A} \to V_{R \times \mathbb{P}^1} \to \mathcal{B} \to 0$$

be the universal short exact sequence on $R \times \mathbb{P}^1$. Recall that when $k \mid d$, we define

$$D_{\mathrm{unb}} = \{ p \in R \mid \mathcal{A}_p \text{ has unbalanced splitting (as a locally free sheaf on } \mathbb{P}^1) \}.$$

So we see that $\alpha \cdot D_{\mathrm{unb}} = 0$ in this case. If $k \nmid d$, then for all p outside a codimension two locus in R, $\mathcal{A}_p$ splits as a direct sum of ℓ_1 copies of $\mathcal{O}_{\mathbb{P}^1}(-d_1)$ and $(k - \ell_1)$ copies of $\mathcal{O}_{\mathbb{P}^1}(-d_1 - 1)$, and the direct sum of the ℓ_1 copies of $\mathcal{O}_{\mathbb{P}^1}(-d_1)$ is a distinguished subsheaf $\mathcal{E}_p$ of $\mathcal{A}_p$. On the curve α this corresponds to the subsheaf

$$E_S = \bigoplus_{i=1}^{\ell_1} A_i^{\vee}(-d_1)$$

of $A_S = E_S \oplus \bigoplus_{i=\ell_1+1}^{k} A_i^{\vee}(-d_1 - 1)$. Recall that D_{unb} in this case is defined by an incidence condition on the directrix $\mathbb{P}(\mathcal{E}_p) \to \mathbb{P}(V)$, hence $\alpha \cdot D_{\mathrm{unb}}$ can be interpreted as $c_2\big(\mathrm{coker}(E_S \to V_S)\big)$, which equals $c_1(E_S)^2 - c_2(E_S)$ by Whitney's formula. Thus

$$\alpha \cdot D_{\mathrm{unb}} = c_1(E_S)^2 - c_2(E_S) = 2d_1\ell_1 \sum_{i=1}^{\ell_1} a_i - d_1(\ell_1 - 1) \sum_{i=1}^{\ell_1} a_i = d_1(\ell_1 + 1) \sum_{i=1}^{\ell_1} a_i.$$

Before turning to the computation of $\alpha \cdot D_{\mathrm{deg}}$, let us elaborate on the construction of D_{deg} itself a little bit more. Our definition of D_{deg} was motivated by the following observation.

Proposition 4.1. *Let $0 \to A \to V_{\mathbb{P}^1} \to B \to 0$ be a short exact sequence of vector bundles on $\mathbb{P}^1$, where as always $V_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1} \otimes V$ is the trivial vector bundle of rank n, $k = \operatorname{rank} A$, $r = \operatorname{rank} B$, and $d = \deg B$. If $d = r$, then the image of $\mathbb{P}(A) \to \mathbb{P}(V)$ is degenerate (i.e. lies in some hyperplane) if and only if B is unbalanced (i.e. not isomorphic to $\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(1)$).*

Proof. There is an induced short exact sequence $0 \to B^{\vee} \to V_{\mathbb{P}^1}^{\vee} \to A^{\vee} \to 0$ of the dual bundles, which in turn induces an exact sequence

$$0 \to H^0(\mathbb{P}^1, B^{\vee}) \to H^0(\mathbb{P}^1, V_{\mathbb{P}^1}^{\vee}) \to H^0(\mathbb{P}^1, A^{\vee})$$

of the spaces of global sections. Identifying $H^0(\mathbb{P}^1, V_{\mathbb{P}^1}^{\vee})$ with $V^{\vee}$, it follows that

$$H^0(\mathbb{P}^1, B^{\vee}) = \{ f \in V^{\vee} \mid f = 0 \text{ on the bundle } A \}.$$

Since $d = r$, B is unbalanced if and only if $H^0(\mathbb{P}^1, B^{\vee}) \neq 0$, namely there exists a nonzero $f \in V^{\vee}$ which is zero on the bundle A, or equivalently, the image of $\mathbb{P}(A) \to \mathbb{P}(V)$ lies in the hyperplane $f = 0$. $\qquad \square$

As mentioned in the Introduction, Proposition 4.1 suggests defining $D_{\deg}$ in general to be "the locus where B is unbalanced". To make this precise, first suppose $r \geq 2$. Recall that the open subset R^0 of R which parametrizes degree d, rank r quotient bundles of $V_{\mathbb{P}^1}$ has an irreducible complement $R \setminus R^0$ of codimension r [8, Corollary 3.3.8]. Since $r \geq 2$, defining $D_{\deg}$ on R is thus equivalent to defining it on R^0. If $r \mid d$, we simply define $D_{\deg}$ on R^0 to be

$$D_{\deg} = \{ p \in R^0 \mid \mathcal{B}_p \text{ has unbalanced splitting (as a locally free sheaf on } \mathbb{P}^1) \}.$$

If $r \nmid d$, set $d = r d_2 + (r - \ell_2)$ for $0 < \ell_2 < r$. Then for all $p \in R^0$ outside a codimension two subset, $\mathcal{B}_p^{\vee}$ splits as a direct sum of ℓ_2 copies of $\mathcal{O}_{\mathbb{P}^1}(-d_2)$ and $(r - \ell_2)$ copies of $\mathcal{O}_{\mathbb{P}^1}(-d_2 - 1)$. In this situation the direct sum of the ℓ_2 copies of $\mathcal{O}_{\mathbb{P}^1}(-d_2)$ is a distinguished subsheaf $\mathcal{F}_p$ of $\mathcal{B}_p^{\vee}$, whose projectivization $\mathbb{P}(\mathcal{F}_p) \to \mathbb{P}(V^{\vee})$ is the directrix of the scroll $\mathbb{P}(\mathcal{B}_p^{\vee}) \to \mathbb{P}(V^{\vee})$. We define $D_{\deg}$ to be the closure in R of the locus of p such that the image of $\mathbb{P}(\mathcal{F}_p) \to \mathbb{P}(V^{\vee})$ meets a fixed $(n - 2 - \ell_2)$-dimensional subspace in $\mathbb{P}(V^{\vee})$.

We now begin to compute $\alpha \cdot D_{\deg}$ in the case $r \geq 2$ and $r \mid d$. Recall that the curve $\alpha \colon C \to R$ is induced by an injective sheaf homomorphism

$$\varphi \colon A_S = \bigoplus_{j=1}^{k} A_j^{\vee}(-m_j) \longrightarrow V_S$$

on $S = C \times \mathbb{P}^1$. Choosing a basis $v_1, \ldots, v_n$ for V and a basis x, y for $H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)\big)$, we can express φ as an $n \times k$ matrix

$$\varphi = \left(\sum_{t=0}^{m_j} s_{ij,t} \, x^t y^{m_j - t} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq k}}$$

where $s_{ij,t} \in H^0(C, A_j)$. We choose the line bundles A_j's to be sufficiently positive and the sections $s_{ij,t}$'s to be sufficiently general so that the cokernel of φ is locally free, which can be done since $r \geq 2$. In other words, we choose the curve $\alpha \colon C \to R$ to lie in R^0, so that the intersection number $\alpha \cdot D_{\mathrm{deg}}$ can be computed geometrically. Let $B_S = \mathrm{coker}\, \varphi$. (Note that, unlike the B_S which defines the curve β in Section 2, the $B_S = \mathrm{coker}\, \varphi$ here is not a direct sum of line bundles on S.) Then $\alpha \cdot D_{\mathrm{deg}}$ equals the number of points $p \in C$ such that $B_{S,p} = B_S|_{\{p\} \times \mathbb{P}^1}$ has unbalanced splitting, or equivalently $H^0\big(\mathbb{P}^1, B_{S,p}^{\vee}(d_2 - 1)\big) \neq 0$ where $d_2 = d/r$. Applying a similar argument as in the proof of Proposition 4.1 to the short exact sequence

$$0 \to A_{S,p} \xrightarrow{\varphi_p} V_{S,p} \to B_{S,p} \to 0$$

of vector bundles on $\{p\} \times \mathbb{P}^1$, we see that

$$H^0\big(\mathbb{P}^1, B_{S,p}^{\vee}(d_2 - 1)\big) = \big\{ f \in H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_2 - 1)\big) \otimes V^{\vee} \mid f = 0 \text{ on } A_{S,p} \big\}.$$

Expressing f as a row vector

$$f = \left(\sum_{t=0}^{d_2 - 1} f_{1,t}\, x^t y^{d_2 - 1 - t}, \ldots, \sum_{t=0}^{d_2 - 1} f_{n,t}\, x^t y^{d_2 - 1 - t} \right)$$

with respect to the dual basis of $v_1, \ldots, v_n$, the condition that $f = 0$ on $A_{S,p}$ is equivalent to the matrix product $f \cdot \varphi_p$ being 0, where φ_p denotes the matrix φ with every $s_{ij,t}$ evaluated at p. For each fixed $p \in C$, $f \cdot \varphi_p = 0$ can be viewed as a system of linear equations in nd_2 unknowns $f_{i,t}$, $1 \leq i \leq n$, $0 \leq t \leq d_2 - 1$. Precisely, for each $j \in \{1, \ldots, k\}$ and each $t \in \{0, \ldots, m_j + d_2 - 1\}$ there is a linear equation

$$\sum_{\substack{t_1 + t_2 = t \\ 1 \leq i \leq n}} s_{ij,t_1}(p) \cdot f_{i,t_2} = 0.$$

The number of equations is thus

$$kd_2 + \sum_{j=1}^{k} m_j = kd_2 + d = kd_2 + rd_2 = nd_2,$$

which is the same as the number of unknowns. Hence the existence of a nonzero solution f is equivalent to the vanishing of the determinant of the coefficient matrix of the linear system. This determinant is a section of the line bundle

$$A_1^{\otimes(m_1 + d_2)} \otimes \cdots \otimes A_k^{\otimes(m_k + d_2)}$$

on C. Therefore

$$
\alpha \cdot D_{\text{deg}} = \sum_{i=1}^{k} a_i(m_i + d_2) = \begin{cases} (d_1 + d_2) \sum_{i=1}^{k} a_i, & \text{if } k \mid d; \\ (d_1 + d_2) \sum_{i=1}^{\ell_1} a_i + (d_1 + d_2 + 1) \sum_{i=\ell_1+1}^{k} a_i, & \text{if } k \nmid d. \end{cases}
$$

This is the formula for $\alpha \cdot D_{\text{deg}}$ when $r \mid d$ and $r \geq 2$. Incidentally, the same formula still holds when $r = 1$. Recall from the Introduction that when $r = 1$, D_{deg} is defined to be $R \setminus R^0$, which is an irreducible divisor by [8, Corollary 3.3.8]. It turns out that its divisor class is equal to D (Proposition 4.2), and one readily checks that the above formula with $r = 1$ (namely $d_2 = d$) gives the same answer as the formula for $\alpha \cdot D$ computed in Section 3.

Proposition 4.2. *If $r = 1$, then $D_{\text{deg}} = R \setminus R^0$ is an irreducible divisor whose divisor class is equal to $D = \pi_{1*}(c_2(\mathcal{B}))$.*

Proof. Since $r = 1$, $\mathcal{B}|_{R^0}$ is a line bundle and hence $c_2(\mathcal{B})|_{R^0} = 0$. It follows that the divisor class $D = \pi_{1*}(c_2(\mathcal{B}))$ can be represented by some divisor supported on $D_{\text{deg}} = R \setminus R^0$, thus $D = cD_{\text{deg}}$ for some integer c since $R \setminus R^0$ is irreducible by [8, Corollary 3.3.8]. Recall from the Introduction that D and Y form a $\mathbb{Z}$-basis for Pic R, hence c must be ± 1. To conclude that $c = 1$, simply observe that $\alpha \cdot D > 0$ by the formula in Section 3, and $\alpha \cdot D_{\text{deg}} \geq 0$ since α does not lie in D_{deg}. $\qquad\square$

Note that if we want to compute $\beta \cdot D_{\text{unb}}$ when $k \mid d$, the situation is completely dual to the computation we just did for $\alpha \cdot D_{\text{deg}}$ when $r \mid d$. So without extra work we have

$$
\beta \cdot D_{\text{unb}} = \sum_{i=1}^{r} b_i(n_i + d_1) = \begin{cases} (d_1 + d_2) \sum_{i=1}^{r} b_i, & \text{if } r \mid d; \\ (d_1 + d_2) \sum_{i=1}^{\ell_2} b_i + (d_1 + d_2 + 1) \sum_{i=\ell_2+1}^{r} b_i, & \text{if } r \nmid d. \end{cases}
$$

Similarly, the computation of $\beta \cdot D_{\text{deg}}$ when $r \geq 2$ is dual to the computation of $\alpha \cdot D_{\text{unb}}$ that was done in the beginning of this section. If $r = 1$ then obviously $\beta \cdot D_{\text{deg}} = 0$, since β lies in R^0 while $D_{\text{deg}} = R \setminus R^0$ in this case. Hence for all $r > 0$ we have

$$
\beta \cdot D_{\text{deg}} = \begin{cases} 0, & \text{if } r \mid d; \\ d_2(\ell_2 + 1) \sum_{i=1}^{\ell_2} b_i, & \text{if } r \nmid d. \end{cases}
$$

Let us summarize the intersection numbers that were computed in this section:

Proposition 4.3. *Let α and β be the test curves defined in Section 2. Let $d_1 = \lfloor d/k \rfloor$, $d_2 = \lfloor d/r \rfloor$, and let ℓ_1, ℓ_2, a_i, and b_i be as defined in Section 2. We have*

(1) $\alpha \cdot D_{\mathrm{unb}} = \begin{cases} 0, & \text{if } k \mid d; \\ d_1(\ell_1 + 1) \sum_{i=1}^{\ell_1} a_i, & \text{if } k \nmid d. \end{cases}$

(2) *If $k \mid d$, then*

$$\beta \cdot D_{\mathrm{unb}} = \begin{cases} (d_1 + d_2) \sum_{i=1}^{r} b_i, & \text{if } r \mid d; \\ (d_1 + d_2) \sum_{i=1}^{\ell_2} b_i + (d_1 + d_2 + 1) \sum_{i=\ell_2+1}^{r} b_i, & \text{if } r \nmid d. \end{cases}$$

(3) *If $r \mid d$, then*

$$\alpha \cdot D_{\mathrm{deg}} = \begin{cases} (d_1 + d_2) \sum_{i=1}^{k} a_i, & \text{if } k \mid d; \\ (d_1 + d_2) \sum_{i=1}^{\ell_1} a_i + (d_1 + d_2 + 1) \sum_{i=\ell_1+1}^{k} a_i, & \text{if } k \nmid d. \end{cases}$$

(4) $\beta \cdot D_{\mathrm{deg}} = \begin{cases} 0, & \text{if } r \mid d; \\ d_2(\ell_2 + 1) \sum_{i=1}^{\ell_2} b_i, & \text{if } r \nmid d. \end{cases}$

We omit the computation of $\beta \cdot D_{\mathrm{unb}}$ when $k \nmid d$ and $\alpha \cdot D_{\mathrm{deg}}$ when $r \nmid d$ because we will not need them.

5. Proof of the theorem when $r > 0$

In this section, we prove Theorem 1.1 when $r > 0$. The case $r = 0$ will be treated in the next section.

First we find the divisor classes of D_{unb} and D_{deg} in terms of the basis D and Y. Starting with D_{unb}, we want to determine the integers e_1 and e_2 such that

$$D_{\mathrm{unb}} = e_1 D + e_2 Y,$$

using the intersection numbers computed in Section 3 and Section 4. We handle the cases $k \mid d$ and $k \nmid d$ separately:

- If $k \nmid d$, we claim that $D_{\text{unb}} = d_1(\ell_1 + 1)\big(-D + (d + d_1 + 1)Y\big)$. This can be seen by intersecting α with $D_{\text{unb}} = e_1 D + e_2 Y$, which gives

$$d_1(\ell_1 + 1) \sum_{i=1}^{\ell_1} a_i = e_1\Big((d + d_1) \sum_{i=1}^{\ell_1} a_i + (d + d_1 + 1) \sum_{i=\ell_1+1}^{k} a_i\Big) + e_2\Big(\sum_{i=1}^{k} a_i\Big)$$

$$= \big(e_1(d + d_1) + e_2\big) \sum_{i=1}^{\ell_1} a_i + \big(e_1(d + d_1 + 1) + e_2\big) \sum_{i=\ell_1+1}^{k} a_i.$$

As this equality holds for all sufficiently large a_i's, one sees that

$$e_1 = -d_1(\ell_1 + 1), \quad e_2 = d_1(\ell_1 + 1)(d + d_1 + 1).$$

- If $k \mid d$, we claim that $D_{\text{unb}} = -D + (d + d_1)Y$. If $r \nmid d$, one can see this by intersecting β with $D_{\text{unb}} = e_1 D + e_2 Y$, which gives

$$(d_1 + d_2) \sum_{i=1}^{\ell_2} b_i + (d_1 + d_2 + 1) \sum_{i=\ell_2+1}^{r} b_i$$

$$= e_1\Big((d - d_2) \sum_{i=1}^{\ell_2} b_i + (d - d_2 - 1) \sum_{i=\ell_2+1}^{r} b_i\Big) + e_2\Big(\sum_{i=1}^{r} b_i\Big)$$

$$= \big(e_1(d - d_2) + e_2\big) \sum_{i=1}^{\ell_2} b_i + \big(e_1(d - d_2 - 1) + e_2\big) \sum_{i=\ell_2+1}^{r} b_i.$$

As this equality holds for all sufficiently large b_i's, one sees that $e_1 = -1$ and $e_2 = d + d_1$. If $r \mid d$, intersecting α with $D_{\text{unb}} = e_1 D + e_2 Y$ yields

$$0 = e_1(d + d_1) \sum_{i=1}^{k} a_i + e_2 \sum_{i=1}^{k} a_i,$$

while intersecting β with $D_{\text{unb}} = e_1 D + e_2 Y$ yields

$$(d_1 + d_2) \sum_{i=1}^{r} b_i = e_1(d - d_2) \sum_{i=1}^{r} b_i + e_2 \sum_{i=1}^{r} b_i.$$

So together they still imply $e_1 = -1$ and $e_2 = d + d_1$.

The divisor class of D_{deg} can be found with similar argument:

- If $r \nmid d$, one sees that

$$D_{\text{deg}} = d_2(\ell_2 + 1)\big(D + (-d + d_2 + 1)Y\big)$$

by intersecting with the curve β.

- If $r \mid d$, one sees that

$$D_{\deg} = D + (-d + d_2)Y$$

by intersecting with the curve α if $k \nmid d$, or intersecting with both α and β if $k \mid d$.

Thus the expressions of the classes of D_{unb} and $D_{\deg}$ in terms of the basis D and Y in Theorem 1.1 are verified.

It remains to prove that D_{unb} and $D_{\deg}$ span the effective cone. The usual strategy to show that an effective divisor lies on the boundary of the effective cone is to check that it has zero intersection with some moving curve. That was the method Coskun-Starr used in [4], where, since they were dealing with the case $d = r$, it is not hard to see that all generic curves are conjugate under the action of $\mathbb{P}GL(V)$ and are thus moving curves ([4, Lemma 2.4]). Unfortunately we were unable to see how to adapt the argument to handle the case $d > r$. So we resort to the following lemma instead, which basically says that since the rank of $\mathrm{Pic}\, R$ is only two, it is enough to *pointwisely* find a curve which has zero intersection with D_{unb} (resp. $D_{\deg}$).

Lemma 5.1. *Suppose two effective divisors D_1 and D_2 on R are linearly independent in $\mathrm{Pic}\, R$. If there is an open subset U of R such that for each point $p \in U$, there exists a curve γ_i (depending on p) in R passing through p and $\gamma_i \cdot D_i = 0$, $i = 1, 2$, then D_1 and D_2 span the effective cone of R.*

Proof. Let E be an arbitrary effective divisor on R and write

$$E = e_1 D_1 + e_2 D_2, \quad e_1, e_2 \in \mathbb{Q}.$$

Pick a point $p \in U$ such that $p \notin E \cup D_2$, and let γ_1 be a curve through p such that $\gamma_1 \cdot D_1 = 0$. Then $\gamma_1 \cdot E \geq 0$ and $\gamma_1 \cdot D_2 > 0$, hence $e_2 \geq 0$. Similarly $e_1 \geq 0$. $\qquad\square$

To apply the lemma, we choose the open subset U of R to be

$$U = \{p \in R^0 \mid \text{Both } \mathcal{A}_p \text{ and } \mathcal{B}_p \text{ have the most balanced splitting types possible}\}.$$

It is not hard to see that given any $p \in U$, there exist curves α and β of the forms described in Section 2 which pass through p and satisfy $\alpha \cdot D_{\mathrm{unb}} = 0$ and $\beta \cdot D_{\deg} = 0$. For example, as done in Section 4, after choosing a basis for V and for $H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)\big)$, we can represent a curve of type α by an $n \times k$ matrix of the form

$$\varphi = \left(\sum_{t=0}^{m_j} s_{ij,t}\, x^t y^{m_j - t} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq k}}$$

where $s_{ij,t} \in H^0(C, A_j)$. A point $p \in U$ can be represented by a similar matrix

$$\left(\sum_{t=0}^{m_j} p_{ij,t}\, x^t y^{m_j - t} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq k}}$$

where $p_{ij,t}$ are constants. In order for the curve φ to pass through p, it is enough that at some point in C, the ratio of the sections $s_{ij,t}$ in each column coincides with the corresponding ratio given by the constants $p_{ij,t}$. And one sees from Proposition 4.3 that $\alpha \cdot D_{\mathrm{unb}}$ is always 0 if $k \mid d$, while if $k \nmid d$ then we take $a_1 = \cdots = a_{\ell_1} = 0$. Although this would require taking the line bundles $A_1, \ldots, A_{\ell_1}$ to be trivial, the other line bundles $A_{\ell_1+1}, \ldots, A_k$ and their sections could still be chosen general enough so that all the previous intersection computations went through. Thus we can conclude from Lemma 5.1 that D_{unb} and D_{deg} span the effective cone of R.

6. Proof of the theorem when $r = 0$

The case $r = 0$ of Theorem 1.1 requires separate treatment for two reasons: the divisor D_{deg} is defined in a totally different manner, and the test curve β defined in Section 2 no longer exists. We do still have the test curve α, as well as the formula for $\alpha \cdot D_{\mathrm{unb}}$ in Proposition 4.3 (1). Let us begin by computing $\alpha \cdot D_{\mathrm{deg}}$. Let

$$\varphi\colon A_S = \bigoplus_{j=1}^{k} A_j^{\vee}(-m_j) \longrightarrow V_S$$

be the injective sheaf homomorphism on $S = C \times \mathbb{P}^1$ that induces the curve $\alpha\colon C \to R$. Choosing a basis $v_1, \ldots, v_n$ for V and a basis x, y for $H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)\big)$, we can express φ as an $n \times n$ matrix

$$\varphi = \left(\sum_{t=0}^{m_j} s_{ij,t}\, x^t y^{m_j - t} \right)_{1 \le i, j \le n}$$

where $s_{ij,t} \in H^0(C, A_j)$. Let $B_S = \operatorname{coker} \varphi$. By the definition of D_{deg}, $\alpha \cdot D_{\mathrm{deg}}$ equals the number of points $p \in C$ such that the support of $B_{S,p} = B_S|_{\{p\} \times \mathbb{P}^1}$ does not consist of d distinct points. The support of $B_{S,p}$ is precisely the set of points on $\mathbb{P}^1$ whose homogeneous coordinates $(x : y)$ satisfy $\det \varphi_p = 0$, where φ_p denotes the matrix φ with every $s_{ij,t}$ evaluated at p. Hence the support of $B_{S,p}$ does not consist of d distinct points if and only if the discriminant of $\det \varphi_p$ vanishes. The discriminant of $\det \varphi$ is a section of the line bundle $(A_1 \otimes \cdots \otimes A_k)^{\otimes 2(d-1)}$ on C. Therefore

$$\alpha \cdot D_{\mathrm{deg}} = 2(d-1) \sum_{i=1}^{k} a_i.$$

Next we will show that, given any point in $R \setminus D_{\mathrm{deg}}$, there exists a curve $\gamma\colon C \to R$ passing through it such that

$$\gamma \cdot D_{\mathrm{deg}} = \gamma \cdot Y = 0.$$

Let B be a degree d, rank 0 quotient sheaf of $V_{\mathbb{P}^1}$ which corresponds to a point in $R \setminus D_{\mathrm{deg}}$, namely $\operatorname{Supp} B$ is a set of d distinct points. We will construct the curve γ by deforming B into a family. In fact, we will only deform the stalk of B at a single

point in $\mathrm{Supp}\, B$. So let p be an arbitrary point in $\mathrm{Supp}\, B$, and denote by $(\mathcal{O}_p, \mathfrak{m}_p)$ the local ring of $\mathbb{P}^1$ at p. Let V_p and B_p be the stalks of the sheaves $V_{\mathbb{P}^1}$ and B at p, respectively, and let A_p be the kernel of $V_p \to B_p$. Since $\mathrm{Supp}\, B$ is a set of d distinct points, $\bigwedge^n A_p$ has colength one in $\bigwedge^n V_p$. Hence there exists an $\mathcal{O}_p$-basis $\nu_1, \ldots, \nu_n$ of V_p such that

$$A_p = \mathfrak{m}_p \nu_1 \oplus \mathcal{O}_p \nu_2 \oplus \cdots \oplus \mathcal{O}_p \nu_n.$$

Let $C = \mathbb{P}^1$ with homogeneous coordinates $(z_0 : z_1)$. Consider the following family of $\mathcal{O}_p$-submodules of V_p over C:

$$A_p(z_0 : z_1) = \begin{cases} \mathfrak{m}_p \nu_1 \oplus \mathcal{O}_p(z_1 \nu_1 + z_0 \nu_2) \oplus \mathcal{O}_p \nu_3 \oplus \cdots \oplus \mathcal{O}_p \nu_n, & \text{if } z_0 \neq 0; \\ \mathfrak{m}_p \nu_2 \oplus \mathcal{O}_p(z_1 \nu_1 + z_0 \nu_2) \oplus \mathcal{O}_p \nu_3 \oplus \cdots \oplus \mathcal{O}_p \nu_n, & \text{if } z_1 \neq 0. \end{cases}$$

This is well-defined, since when both z_0 and z_1 are nonzero, the two expressions give the same submodule. Let $B_p(z_0 : z_1) = V_p/A_p(z_0 : z_1)$, and let $B(z_0 : z_1)$ be the sheaf on $\mathbb{P}^1$ whose stalk at p is $B_p(z_0 : z_1)$, and at every other point is just the same as the stalk of B. Then $B(z_0 : z_1)$ induces a curve $\gamma : C \to R$ which maps $(1 : 0) \in C$ to the point in R corresponding to B. Note that $\mathrm{Supp}\, B(z_0 : z_1) = \mathrm{Supp}\, B$ for all $(z_0 : z_1)$. This implies that γ does not meet D_{deg}, hence $\gamma \cdot D_{\mathrm{deg}} = 0$. It also implies that, viewed as a sheaf on the surface $S = C \times \mathbb{P}^1$, $B(z_0 : z_1)$ (and hence also its first Chern class) is completely supported on $C \times \mathrm{Supp}\, B$, thus $\gamma \cdot Y = 0$.

We now proceed to find the divisor classes of D_{unb} and D_{deg} in terms of the basis D and Y. If $k \nmid d$, this can be done using the curve α alone: the class of D_{unb} has been worked out in Section 5 to be

$$D_{\mathrm{unb}} = d_1(\ell_1 + 1)\big(-D + (d + d_1 + 1)Y\big).$$

As for D_{deg}, writing $D_{\mathrm{deg}} = e_1 D + e_2 Y$ and intersecting with α gives

$$2(d-1) \sum_{i=1}^{k} a_i = e_1 \Big((d + d_1) \sum_{i=1}^{\ell_1} a_i + (d + d_1 + 1) \sum_{i=\ell_1+1}^{k} a_i \Big) + e_2 \Big(\sum_{i=1}^{k} a_i \Big)$$

$$= \big(e_1(d + d_1) + e_2 \big) \sum_{i=1}^{\ell_1} a_i + \big(e_1(d + d_1 + 1) + e_2 \big) \sum_{i=\ell_1+1}^{k} a_i.$$

It follows that $e_1 = 0$, $e_2 = 2(d-1)$, and hence $D_{\mathrm{deg}} = 2(d-1)Y$. If $k \mid d$, we can still deduce that $D_{\mathrm{deg}} = 2(d-1)Y$ with the extra help from the new curve γ, for intersecting α with $D_{\mathrm{deg}} = e_1 D + e_2 Y$ now gives

$$2(d-1) \sum_{i=1}^{k} a_i = e_1(d + d_1) \sum_{i=1}^{k} a_i + e_2 \sum_{i=1}^{k} a_i = \big(e_1(d + d_1) + e_2 \big) \sum_{i=1}^{k} a_i,$$

which is not enough to determine e_1 and e_2. So we need the additional information from intersecting γ with $D_{\mathrm{deg}} = e_1 D + e_2 Y$, which turns out to be $e_1 = 0$ since

$\gamma \cdot D_{\mathrm{deg}} = \gamma \cdot Y = 0$. Hence $e_2 = 2(d-1)$, and $D_{\mathrm{deg}} = 2(d-1)Y$ still holds. As for D_{unb}, writing $D_{\mathrm{unb}} = e_1 D + e_2 Y$ and intersecting with α gives

$$0 = e_1(d+d_1)\sum_{i=1}^{k} a_i + e_2 \sum_{i=1}^{k} a_i = \big(e_1(d+d_1) + e_2\big) \sum_{i=1}^{k} a_i,$$

which implies that

$$D_{\mathrm{unb}} = c_1\big(-D + (d+d_1)Y\big)$$

for some nonzero constant c_1. Unfortunately since we do not know how to compute $\gamma \cdot D_{\mathrm{unb}}$ and $\gamma \cdot D$, we do not know the value of c_1. However we can at least see that $c_1 > 0$ once we know that D_{unb} and D_{deg} span the effective cone, since (the closure of) the effective cone has to contain the nef cone (cf. Figure 1 right).

It remains to show that D_{unb} and D_{deg} span the effective cone. For this we use Lemma 5.1, taking the open subset U of R to be

$$U = \{p \in R \mid \mathcal{A}_p \text{ has the most balanced splitting type possible}\} \setminus D_{\mathrm{deg}}.$$

As we have seen, given any $p \in U$, there exist curves α and γ which pass through p and satisfy $\alpha \cdot D_{\mathrm{unb}} = 0$ and $\gamma \cdot D_{\mathrm{deg}} = 0$. Hence D_{unb} and D_{deg} span the effective cone of R by Lemma 5.1.

References

[1] A. Bertram, *Towards a Schubert calculus for maps from a Riemann surface to a Grassmannian*, Internat. J. Math. **5** (1994), 811–825.

[2] A. Bertram, *Quantum Schubert calculus*, Adv. Math. **128** (1997), 289–305.

[3] A. Bertram, G. Daskalopoulos and R. Wentworth, *Gromov invariants for holomorphic maps from Riemann surfaces to Grasmannians*, J. Amer. Math. Soc. **9** (1996), 529–571

[4] I. Coskun and J. Starr, *Divisors on the space of maps to Grassmannians*, Int. Math. Res. Notices, vol. 2006, Article ID 35273, 25 pages, 2006.

[5] C. Ramirez, *The degree of the variety of rational ruled surfaces and Gromov-Witten invariants*, Trans. Amer. Math. Soc. **358** (2006), 11–24.

[6] C. Ramirez, *On a stratification of the Kontsevich moduli space $\overline{M}_{0,n}(G(2,4),d)$ and enumerative geometry*, J. Pure Appl. Algebr. **213** (2009), 857–868.

[7] M. Ravi, J. Rosenthal, and X. Wang, *Degree of the generalized Plücker embedding of a Quot scheme and quantum cohomology*, Math. Ann. **311** (1998), 11–26.

[8] Yijun Shao, *A compactification of the space of algebraic maps from $\mathbb{P}^1$ to a Grassmannian*, Ph.D. Thesis, University of Arizona.

[9] S. Strømme, *On parametrized rational curves in Grassmann varieties*, Lecture Notes in Mathematics, Volume 1266, Springer, Berlin, 1987, 251–272.

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